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EXIT PROPERTIES OF STOCHASTIC PROCESSES WITH STATIONARY INDEPENDENT INCREMENTS

BY

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ABSTRACT. Let $\{X_t, t \geq 0\}$ be a real stochastic process with stationary independent increments. For $x > 0$, define the exit time T_x from the interval $(-\infty, x]$ by $T_x = \inf\{t > 0: X_t > x\}$. A reasonably complete solution is given to the problem of deciding precisely when $P^0\{X_{T_x} = x\} > 0$ and precisely when $P^0\{X_{T_x} = x\} = 0$. The solution is given in terms of parameters appearing in the Lévy formula for the characteristic function of X_t. A few applications of this result are discussed.

1. **Introduction.** Let $X = \{X_t, t \geq 0\}$ be a real valued stochastic process with stationary independent increments. Then $E^0 e^{iuX_t} = \exp\{t\psi(u)\}$, where

$$\psi(u) = iau - (\sigma^2/2)u^2 + \int [e^{iux} - 1 - iux/(1 + x^2)]\nu(dx). \tag{1.1}$$

The measure ν is called the Lévy measure, and ψ is called the exponent of the process X. If $\sigma^2 > 0$, X is said to have a Gaussian component. If $\int_{-1}^{1} |x|\nu(dx) < \infty$, then as is customary we will assume the exponent written in the form

$$\psi(u) = ia'u - (\sigma^2/2)u^2 + \int [e^{iux} - 1]\nu(dx). \tag{1.2}$$

If in (1.2), $\sigma^2 = 0$, $\nu\{(-\infty, 0)\} = 0$, $a' \geq 0$ then the corresponding process has increasing paths and is called a subordinator; the constant a' is then called the drift.

For $x > 0$, define $T_x = \inf\{t > 0: X_t > x\}$. The basic problem of this paper is to decide in terms of the exponent when $P^0\{X_{T_x} = x\} > 0$, and when $P^0\{X_{T_x} = x\} = 0$. Thus in the latter case X jumps across the boundary on its first exit from $(-\infty, x]$. In the former case we will say for brevity that X has continuous (upward) passages across the level x. Besides their intrinsic interest as descriptions of sample function behavior, results of this nature are often

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tools in proving other properties of interest; see Theorem 1.1 below and the paragraph at the end of this section.

The problem is obviously trivial unless the process hits points and has upward jumps. Therefore we assume throughout that

$$\nu\{(0, \infty)\} > 0 \tag{1.3}$$

and, if $H_x = \inf\{t > 0\colon X_t = x\}$,

$$P^0\{H_x < \infty\} > 0, \qquad x > 0. \tag{1.4}$$

Precise conditions under which (1.4) holds have been given by Kesten [11]; see also [6]. Moreover, in order to avoid discussing the uninteresting case where the process has only a finite number of jumps in every finite time interval, we further assume throughout that

$$\nu\{(-\infty, \infty)\} = \infty. \tag{1.5}$$

From assumptions (1.3) and (1.5) it follows that

$$P^0\{X_{T_x -} = x < X_{T_x}\} = P^0\{X_{T_x -} < x = X_{T_x}\} = 0, \tag{1.6}$$

so that with probability one a given path either jumps across x strictly at time T_x, or else hits x in a continuous manner. This result is well known; proofs may be found in [11], [13], [15]. It also turns out under the present hypotheses (see Corollary 3.1) that either $P^0\{X_{T_x} = x\} > 0$ for all $x > 0$ or $P^0\{X_{T_x} = x\} = 0$ for all $x > 0$. Finally, under the present hypotheses, Proposition 2.1 guarantees that $P\{X_{T_x} > x\} > 0$ for all $x > 0$.

With these preliminaries aside we may now describe the main results of the paper. It is convenient to distinguish the following possible conditions on the exponent:

$$
\begin{array}{lllll}
 & \text{(a)} & \sigma^2 = 0, & \int_{-1}^{1} |x|\nu(dx) < \infty, & a' > 0, \\
 & \text{(b)} & \sigma^2 = 0, & \int_{-1}^{1} |x|\nu(dx) < \infty, & a' < 0, \\
 & \text{(c)} & \sigma^2 \neq 0, & & \\
(1.7) & \text{(d)} & \sigma^2 = 0, & \int_{-1}^{0} |x|\nu(dx) = \infty, & \int_{0}^{1} x\nu(dx) < \infty, \\
 & \text{(e)} & \sigma^2 = 0, & \int_{-1}^{0} |x|\nu(dx) < \infty, & \int_{0}^{1} x\nu(dx) = \infty, \\
 & \text{(f)} & \sigma^2 = 0, & \int_{-1}^{0} |x|\nu(dx) = \int_{0}^{1} x\nu(dx) = \infty. &
\end{array}
$$

These are the only possibilities for which (1.4) can hold (see [11]). In cases (b) and (e) we show that $P^0\{X_{T_x} = x\} = 0$ for all $x > 0$, while in cases (a), (c), and (d), $P^0\{X_{T_x} = x\} > 0$ for all $x > 0$. In case (f) both possibilities can occur. An analytic criterion in terms of ψ is given in Theorem 3.2:

$$P^0\{X_{T_x} = x\} > 0 \quad \text{for all } x > 0 \text{ if and only if} \tag{1.8}$$
$$\int_0^1 K(y)\nu\{(y, 1)\}dy < \infty,$$

where

$$K(y) = \lim_{\lambda \downarrow 0} \int_{-\infty}^{\infty} [1 - \cos uy]\mathrm{Re}\{(\lambda - \psi(u))^{-1}\}du.$$

This criterion can be used to show that if $\nu\{(-\infty, -x)\} = O(\nu\{(x, \infty)\})$ as $x \downarrow 0$, then $P^0\{X_{T_x} = x\} = 0$ for all $x > 0$ (see Theorem 3.6). In particular, the conclusion of the preceding sentence holds for all symmetric processes and all stable processes whose Lévy measure is not concentrated on $(-\infty, 0)$. On the other hand the criterion (1.8) can be used to show that (in case (f)) if ν restricted to $(0, \infty)$ is rather smaller than ν restricted to $(-\infty, 0)$, then $P^0\{X_{T_x} = x\} > 0$ for all $x > 0$ (see Theorem 3.4 for the precise statement).

The following is a simple but amusing application of some of these ideas. For $x > 0$, define $T_{-x} = \inf\{t > 0: X_t < -x\}$.

Theorem 1.1. *Let X be a real process with stationary, independent increments such that*

(i) $P^0\{H_x < \infty\} > 0$ *for all* x;

(ii) $P^0\{T_x < \infty\} = P^0\{T_{-x} < \infty\} = 1$ *for all* $x > 0$;

(iii) $P^0\{X_{T_x} = x\} = P^0\{X_{T_{-x}} = -x\} = 0$ *for all* $x > 0$.

If $x \neq 0$, then starting from 0, X jumps across the point x infinitely often before hitting it.

The proof, of course, is a simple application of the strong Markov property. Symmetric processes that hit points and the stable processes of index $2 > \alpha > 1$ (with $\nu\{(-\infty, 0)\} > 0$, $\nu\{(0, \infty)\} > 0$) satisfy the hypotheses of this theorem.

The main problem of this paper is a special case of a much more general problem that may be formulated as follows. Let X be a process with stationary independent increments with values in R^n. Let B be a closed set in R^n which is a positive distance from 0. Let $T_B = \inf\{t > 0: X_t \in B\}$. The problem is then to determine when

$$P^0\{X_{T_B} \in \partial B,\ T_B < \infty\} = 0. \tag{1.9}$$

In spite of its intrinsic interest in the description of sample function behavior, and in spite of the fact that (1.9) appears as a hypothesis in at least one important paper [10], very little seems to be known regarding the solution of this problem.

The organization of the paper is as follows. §2 contains the proof of the rather simple cases (a) and (b), while §3 treats the more difficult cases (c), (d), (e), (f). Notation and terminology belonging to the theory of Markov processes follow that of [3]. As usual, the process X is assumed (without loss of generality) to be a Hunt process. The notation P^x is the measure for the process starting at x; if $x = 0$, then the superscript will usually be omitted.

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2. **The simple processes: cases (a) and (b).** This section treats the relatively simple cases (a) and (b) of (1.7). The proofs below could in certain cases be shortened by introducing considerations involving local time (as in §3); however, it was felt that appeal to such a sophisticated concept was a bit inelegant in such simple situations as the ones discussed in this section.

The reader wishing to go directly to §3 need read only Proposition 2.5 of this section. *Recall that assumptions* (1.3), (1.4), (1.5) *are in effect throughout; these assumptions will not always appear explicitly in the hypotheses.*

Proposition 2.1. *Let* $\delta > 0$ *and* $x > 0$. *Then* $P\{X_{T_x} \in (x, x + \delta)\} > 0$.

Proof. It is obviously enough to prove the result for δ small. Suppose first that $\nu\{(0, \infty)\} = \infty$. Let $a_1 > 0$ be a point of increase for ν such that $a_1 < x$. Since $\nu\{(0, \infty)\} = \infty$, there are then positive numbers $a_1 \geq a_2 \geq \cdots \geq a_n > 0$ which are points of increase of ν such that $a_1 + \cdots + a_n$ is within ϵ of $x + a_1/4$; here $0 < \epsilon < a_1/12$. Let I_j be a small open interval about a_j such that if $x_j \in I_j$ then $x_1 + \cdots + x_n$ is within 2ϵ of $x + a_1/4$. Let ν_j be ν restricted to I_j and let $x^1, \cdots, x^n$ be the independent compound Poisson processes corresponding to $\nu_1, \cdots, \nu_n$ ($Ee^{iuX_t^j} = \exp\{-t\psi_j(u)\}$ where $\psi_j(u) = \int_{I_j} [1 - e^{iux}]\nu(dx)$). Then we can write $X = X^1 + \cdots + X^n + Y$ where all the processes in this representation are independent. Let $\Delta > 0$ be chosen so small that $P\{\sup_{0 \leq t \leq \Delta} |Y_t| \leq \epsilon\} > 0$. Then using the independence of $X^1, \cdots, X^n$ and Y we have with positive probability: $X^1, \cdots, X^n$ will jump exactly once in the time interval $[0, \Delta]$, X^1 will jump last, and Y will remain less than ϵ throughout $[0, \Delta]$. The result of this will put us within 3ϵ of $x + a_1/4$, and moreover, up until the time X^1 jumps (in this scheme) we will not yet have exceeded x. Thus T_x coincides here with the time of the jump of X^1, and $x < X_{T_x} < (a_1) + x$. Since a_1 may be chosen as small as desired, this completes the proof in case $\nu\{(0, \infty)\} = \infty$.

If $\nu\{(0, \infty)\} < \infty$, then by (1.5), $\nu\{(-\infty, 0)\} = \infty$, and we can proceed as follows. Let $a > 0$ be a point of increase of ν. Let n be a positive integer such that $x + a/4 - na < 0$, and let $\epsilon > 0$ be a small number. Using the argument above one can with positive probability get within ϵ of $x + a/4 - na$ before there are any positive jumps at all and before the process rises above x. Having got here with positive probability we can then experience (with positive probability) n successive positive jumps of approximately size a before the rest of the process changes by more than ϵ. This will land us within 3ϵ of $x + a/4$ with positive probability; appropriate choice of ϵ completes the proof.

Proposition 2.2. *Let* $h(x) = P\{X_{T_x} > x\}$. *Then* $h(x) > 0$ *for every* $x > 0$ *and the map* $x \longrightarrow h(x)$ *of* $(0, \infty)$ *to* $[0, 1]$ *is lower semicontinuous.*

Proof. The positivity of h is immediate from Proposition 2.1. Let $I\{A\}$ be the indicator function of the set A. By (1.6) $\lim_{x\to z} I\{\omega: X_{T_x} > x > X_{T_x-}\} = I\{\omega: X_{T_z} > z > X_{T_z-}\}$ on $\{\omega: X_{T_z} > z > X_{T_z-}\}$, implying that $\lim_{x\to z}\inf h(x) \geq h(z)$, as desired.

Remark. Continuity of h is proved in §3.

Corollary 2.1. *If* $0 < a < b$, *then* $\{h(z): a \leq z \leq b\}$ *is bounded away from* 0.

The next lemma essentially says that if there are continuous crossings of a level x, then the closer to x one starts, the more likely is one to cross x continuously. The definitive version of this lemma is in Corollary 3.2.

Proposition 2.3. *Consider the following statements*:

(a) $P\{X_{T_x} = x\} > 0$ *for some* $x > 0$.

(b) $\lim_{y\downarrow 0}\inf P\{X_{T_y} > y\} = 0$.

Then (a) *implies* (b).

Proof. Abbreviate $T = T_x$, $T_n = \inf\{t > 0: X_t > x - (1/n)\}$, and set $A_n = \{T_n < T\} = \{X_{T_n} < x\}$, $A = \bigcap A_n$. Note that $A_n \supset A_{n+1}$. If (a) is assumed then $P(A) > 0$ since $\{X_{T_x} = x\} \subset A_n$. Then $0 = P\{X_{T-} < x; A\} = \lim_{n\to\infty} P\{X_{T-} < x; A_n\}$ $= \lim_{n\to\infty} \int_{A_n} P^{X_{T_n}}\{X_{T-} < x\}dP \geq \lim_{n\to\infty} \int_A P^{X_{T_n}}\{X_{T-} < x\}dP$. Hence $P^{X_{T_n}}\{X_{T-} < x\} \longrightarrow 0$ in probability on A as $n \longrightarrow \infty$. A subsequence therefore converges to 0 a.s. (P^0) on A, so that for some sequence $\{y_n\}$, $y_n \uparrow x$, $\lim_{n\to\infty} P^{y_n}\{X_{T-} < x\} = 0$. Statement (b) now follows from spatial homogeneity.

Proposition 2.3 permits an easy proof of the stable case. In case X is symmetric stable, this was proved by S. Watanabe by a different method [22].

Corollary 2.2. *Let* X *be a stable process of index* α, $1 < \alpha < 2$, *with* $\nu\{(0, \infty)\} > 0$. *Then* $P\{X_{T_x} > x\} = 1$ *for every* $x > 0$.

Proof. By the scaling property, for any $r > 0$ the process $\{r^{1/\alpha}X(rt)\}$ has the same distribution as the process $\{X(t)\}$. Then, since the property of jumping across a given level at the first passage time is independent of contractions of the time scale, we have

$$\begin{aligned} &P\{X(t) \text{ exceeds } x \text{ for the first time by jumping}\} \\ &\quad = P\{r^{1/\alpha} X(rt) \text{ exceeds } x \text{ for first time by jumping}\} \\ &\quad = P\{X(rt) \text{ exceeds } r^{-1/\alpha} x \text{ for first time by jumping}\} \\ &\quad = P\{X(t) \text{ exceeds } r^{-1/\alpha} x \text{ for first time by jumping}\}. \end{aligned}$$

Choosing $r = x^{\alpha}$, we see from this calculation and Proposition 2.1 that $P\{X_{T_x} > x\} = b(x)$ is positive and constant for $x > 0$. Hence (b) in Proposition 2.3 cannot hold, so $P\{X_{T_x} = x\} = 0$. Since $P\{T_x < \infty\} = 1$ in the present case, $P\{X_{T_x} > x\} = 1$ as desired.

We can now settle case (b) of (1.7).

Theorem 2.1. *Suppose* $\nu\{(-\infty, \infty)\} = \infty$, $\nu\{(0, \infty)\} > 0$, $\int_{-1}^{1} |x|\nu(dx) < \infty$, *and* $a' < 0$. *Then* $P\{X_{T_x} = x\} = 0$ *for every* $x > 0$.

Proof. Under the present hypotheses, $\lim_{t\downarrow 0} X_t/t = a'$ a. s., so that X is negative in an initial random time interval. (This fact is well known; see, for example, Shtatland [21].) Consequently there is a number $b > 0$ such that if $T_{-b} = \inf\{t > 0\colon X_t < -b\}$ and $T_0 = \inf\{t > 0\colon X_t > 0\}$ then $P\{T_{-b} < T_0\} > 0$. Let $\{F_t, t \geq 0\}$ be the usual sigma fields associated with the Markov process $\{X_t\}$ (see [3]); and let $I\{A\}$ denote the indicator function of a set A. Choose $n > 0$ so large that $P\{T_{-b} < T_0, X_{T_{-b}} \in [-n, -b]\} > 0$. Then

$$\begin{aligned} P\{X_{T_x} > x\} &\geq P\{T_{-b} < T_x, X_{T_x} > x\} \\ &\geq P\{T_{-b} < T_0, X_{T_{-b}} \in [-n, -b], X_{T_x} > x\} \\ &= EE^{F_{T_{-b}}} I\{T_{-b} < T_0, X_{T_{-b}} \in [-n, -b]\} I\{X_{T_x} > x\} \\ &= EI\{T_{-b} < T_0, X_{T_{-b}} \in [-n, -b]\} P^{X_{T_{-b}}}\{X_{T_x} > x\}. \end{aligned}$$

It follows from Proposition 2.2 that the function of y given by $y \to P^{-y}\{X_{T_x} > x\} = P^0\{X_{T_{x+y}} > x + y\}$ is bounded away from 0 for y restricted to the interval $[b, n]$. Moreover if x is restricted to $[0, 1]$ then the same proposition allows us to choose this positive lower bound independently of x. Letting B be this positive lower bound, we see that for $x \in (0, 1]$,

$$P\{X_{T_x} > x\} \geq BP\{T_{-b} < T_0, X_{T_{-b}} \in [-n, -b]\} > 0.$$

It follows that statement (b) of Proposition 2.3 cannot hold, and this proves the theorem.

The next proposition can be strengthened (see Corollary 3.4) but it is enough for present purposes.

Proposition 2.4. *Let X have Lévy measure ν. Let (a, b) be an open interval containing 0, and let X^1 be the truncated process obtained from X by removing all the jumps having size in $(-\infty, a] \cup [b, \infty)$ (so that $E\exp\{iuX_t^1\} = \exp\{\psi_1(u)\}$ with $\psi_1(u) = \psi(u) - \int_{x\notin(a,b)} [e^{iux} - 1]\nu(dx)$). Let $T_x^1 = \inf\{t > 0\colon X_t^1 > x\}$ and let $\delta > 0$. If $P\{X^1_{T_x^1} = x\} > \delta$ for all sufficiently small x, then $P\{X_{T_x} = x\} > \delta/2$ for all small x.*

Proof. This is true for case (b) by Theorem 2.1. In all other cases, 0 is regular for $(0, \infty)$ and so $\lim_{x\downarrow 0} T_x = 0 = \lim_{x\downarrow 0} T_x^1$ a.s. (see Rogozin [20]; 0 regular for $(0, \infty)$ means $P^0\{T_0 > 0\} = 0$). However, using the Ito construction of X and X^1, one sees that the paths of X^1 agree with those of X in an initial time interval and so for all sufficiently small x (how small depends on ω) the paths of X^1 and X will agree up to time T_x.

The following formula for $P\{X_{T_x} > x\}$ will be used several times. As usual, $I_A(x) = 1$ if $x \in A$, $= 0$ if $x \notin A$.

Proposition 2.5.

$$P\{X_{T_x} > x\} = E\int_{y=0}^{\infty}\int_{s=0}^{T_x} I_{(x-y,x)}(X_s)\,ds\,\nu(dy). \tag{2.1}$$

Proof. Let $f(u, v)$ be a nonnegative Borel function from $R \times R$ to R that vanishes on the diagonal. According to the theory of Lévy systems (see S. Watanabe [23]; a fairly simple proof for the present situation may be found in [14]):

$$\sum_{s\le t} f(X_{s-}, X_s) - \int_{s=0}^{t}\int_{y\in R} f(X_s, X_s + y)\nu(dy)\,ds$$

is a martingale of mean 0, assuming expectations exist. Define $f_n(u, v) = 1$ if $u < x - n^{-1} < x < x + n^{-1} < v$, $f_n(u, v) = 0$ otherwise, and $f(u, v) = 1$ if $u < x < v$, $f(u, v) = 0$ otherwise. Let $M > 0$. By the optional sampling theorem,

$$E\sum_{s\le T_x\wedge M} f_n(X_{s-}, X_s) = E\int_0^{T_x\wedge M}\int_{y=0}^{\infty} f_n(X_s, X_s + y)\nu(dy)\,ds.$$

Let $n\uparrow\infty$ and then $M\uparrow\infty$. Using the monotone convergence theorem one obtains

$$E\sum_{s\le T_x} f(X_{s-}, X_s) = E\int_0^{T_x}\int_{y=0}^{\infty} f(X_s, X_s + y)\nu(dy)\,ds$$

from which (2.1) follows.

Let us now take care of case (a).

Theorem 2.2. *Assume* $\nu\{(-\infty, \infty)\} = \infty$, $\nu\{(0, \infty)\} > 0$, $\int_{-1}^{1} |x|\nu(dx) < \infty$, *and* $a' > 0$. *Then* $P\{X_{T_x} = x\} > 0$ *for all* $x > 0$.

Proof. The process X satisfying the hypotheses above may be written $X_t = a' t + X_t' - X_t''$, where X' and X'' are independent subordinators. We first prove that $P\{X_{T_x} = x\} > \epsilon > 0$ for all sufficiently small x. According to Proposition 2.4 we may assume that there are no jumps bigger then 1 in magnitude (i.e., $\nu\{[x, \infty)\} = \nu\{(-\infty, -x]\} = 0$ for all $x > 1$) so that X_t', X_t'' have finite expectations, and by the same proposition we may assume that $EX_t'' = ct$, where $0 \le c < a'$, by truncating ν on $(-\infty, 0)$ even further if necessary. Then according to Proposition 2.5:

$$P\{X_{T_x} > x\} = \int_{y=0}^{1} E\int_{s=0}^{T_x} I_{(x-y,x)}(X_s)\, ds\nu(dy).$$

But

$$E\int_0^{T_x} I_{(x-y,x)}(X_s)\,ds \le E(T_x - T_{x-y})$$
$$\le ET_y \quad \text{if } x \ge y$$
$$E\int_0^{T_x} I_{(x-y,x)}(X_s)\,ds \le ET_x \quad \text{if } x < y \le 1,$$

so that

$$P\{X_{T_x} > x\} \le \int_0^x ET_y\nu(dy) + ET_x\int_x^1 \nu(dx). \tag{2.2}$$

Let $S_x = \inf\{t > 0: a't - X_t'' > x\}$. Since X' has nonnegative increasing paths, $S_x \ge T_x$. Let n be a positive integer. Since the process $a't - X_t''$ has no upward jumps

$$E\{a'S_x \wedge n - X''_{S_x\wedge n}\} \le x,$$

so $a'ES_x \wedge n \le x + EX''_{S_x\wedge n} = x + cE(S_x \wedge n)$ (where the equality comes from the optional sampling theorem). Hence $(a' - c)ES_x \le x$, implying that $ET_x \le x/(a' - c)$. Therefore by (2.2)

$$P\{X_{T_x} > x\} \le K\int_0^x y\nu(dy) + Kx\int_x^1 \nu(dx) \qquad (K = (a' - c)^{-1}). \tag{2.3}$$

Since $\int_{-1}^{1} |x|\nu(dx) < \infty$, the right side of (2.3) goes to 0 as $x \downarrow 0$, so that $P\{X_{T_x} = x\} > 0$ for all sufficiently small x, say $x \le \delta$. To complete the proof, we verify that $P\{X_{T_x} = x\} > 0$ for all $x > 0$. Let $x > \delta$ and let $R = \inf\{t > 0: X_t > x - (\delta/2)\}$. By Proposition 2.1, $P\{X_R \in (x - (\delta/2), x - (\delta/4))\} > 0$. The theorem for general x now follows by the strong Markov property and the fact that if holds for all $x \le \delta$.

We close this section by indicating an alternative approach to case (a). This approach, however, depends on local time. A perusal of Kesten's Lemma 2.4 [11, p. 16] and his calculation on p. 21 of [11] reveals that $P\{X_{T_x} = x\} > 0$ for x in a set of positive measure. Appeal to Corollary 3.1 in the next section finishes the proof.

3. **The cases (c), (d), (e), (f).** Let $X = \{X_t, t \geq 0\}$ be a real process with stationary independent increments having 0 regular for $(0, \infty)$. Let $X_t^* = \sup_{s \leq t} X_s$. Let $Q = \{Q_t, t \geq 0\}$ be the local time at 0 for the Markov process $\{X_t^* - X_t\}$ (that Q exists follows from the assumption that 0 is regular for $(0, \infty)$). Let

$$\tau_t = \inf\{s \geq 0: Q_s > t\} \tag{3.1}$$

be the right continuous inverse of Q. Fristedt has shown [8] that the process $Y = \{Y_t, t \geq 0\}$ defined by

$$Y_t = X^*_{\tau_t} \tag{3.2}$$

is a subordinator. Since 0 is regular for $(0, \infty)$, (see [20]) $T_x = \inf\{t \geq 0: X_t > x\} = \inf\{t \geq 0: X_t \geq x\}$ a.s. and since Y has strictly increasing paths, $S_x \equiv \inf\{t > 0: Y_t > x\} = \inf\{t > 0: Y_t \geq x\}$ a.s.

Proposition 3.1. *Let X be a process with stationary independent increments having 0 regular for $(0, \infty)$. Then for each $x > 0$*

$$Y_{S_x} = X_{T_x} \quad a.s. \tag{3.3}$$

Proof. From the definitions it follows that

$$Y_t < x \quad \text{if and only if} \quad \tau_t < T_x. \tag{3.4}$$

From (3.4) and the fact that τ_t is strictly increasing (up to its terminal time) and has the continuous inverse Q_t, we see that

$$S_x = \inf\{t > 0: Y_t \geq x\} = \inf\{t > 0: \tau_t \geq T_x\} = Q \circ T_x \quad \text{a.s.} \tag{3.5}$$

Moreover, if t is a point of right increase for $Q_t(\omega)$,

$$\tau \circ Q_t(\omega) = t. \tag{3.6}$$

Since $P\{T_x$ is a point of right increase of $Q\} = P\{T_x < \infty\}$ by V.3.5 of [3] (essentially), it follows from (3.6) that $Y_{S_x} = X_{\tau \circ Q \circ T_x} = X_{T_x}$ a.s.

Corollary 3.1. *Let X satisfy the hypotheses of Proposition 3.1 and let $x > 0$. Then $P\{X_{T_x} = x\} > 0$ if and only if the subordinator Y has positive drift.*

Moreover, either $P\{X_{T_x} = x\} > 0$ *for all* $x > 0$ *or* $P\{X_{T_x} = x\} = 0$ *for all* $x > 0$.

Proof. Immediate from (3.3) and Theorem 1c of Kesten's monograph [11]. Define $h(x) = P\{X_{T_x} > x\}$, $g(x) = P\{X_{T_x} = x\}$.

Corollary 3.2. *Let X be any real process with stationary independent increments satisfying either* (1.5) *or* $\sigma^2 > 0$. *Then g and h are continuous functions on* $(0, \infty)$. *If* $g(x) > 0$ *for some* $x > 0$, *then* $\lim_{x\downarrow 0} g(x) = 1$.

Proof. If the process does not hit points then g is trivially continuous. The case where X hits points and 0 is not regular for $(0, \infty)$ is treated in Theorem 2.1, which shows that g is identically 0 in this case. Finally, if 0 is regular for $(0, \infty)$ then $g(x) = P\{X_{T_x} = x\} = P\{Y_{S_x} = x\}$; in this case g is either identically 0 or positive, continuous, and with the announced limit at 0 by Proposition 6, p. 120 of Kesten's monograph [11]. Hence in any case, g is continuous. Since $g(x) + h(x) = P\{T_x < \infty\}$, one proves h continuous by proving $x \to P\{T_x < \infty\}$ continuous. If $x \downarrow z$, then $T_x \downarrow T_z$ so $P\{T_x < \infty\}$ is right continuous and decreases as x increases. If $\delta > 0$, then

$$P\{T_x < \infty\} \geq P\{T_\delta < \infty\} P\{T_{x-\delta} < \infty\};$$

so if 0 is regular for $(0, \infty)$, then $\lim_{\delta\downarrow 0} P\{T_\delta < \infty\} = 1$, implying in this case that $P\{T_x < \infty\}$ is also left continuous. In the only remaining cases, $X_{T_x -} < x < X_{T_x}$ on $\{T_x < \infty\}$, which can be shown to imply left continuity here as well.

Remark. From Theorem 1 of [10], one may deduce that h is excessive for the subprocess obtained by killing X upon first leaving $(-\infty, x)$. This observation provides an alternative route to proving some of the regularity properties of h.

The next two corollaries give refinements of Propositions 2.3 and 2.4. In each case the proof is nearly immediate.

Corollary 3.3. *The following statements are equivalent*:

(a) $P\{X_{T_x} = x\} > 0$ *for some (hence all)* $x > 0$;

(b) $\lim_{y\downarrow 0} P\{X_{T_y} > y\} = 0$.

Corollary 3.4. *Let* X^1 *be obtained from X by truncation as in Proposition* 2.4. *Then*

$$P\{X_{T_x} = x\} > 0 \text{ for some (hence all) } x > 0 \text{ if and only if}$$

$$P\{X^1_{T^1_x} = x\} > 0 \text{ for some (hence all) } x > 0.$$

This corollary states that the manner in which a process passes over a given level is independent of the large jumps of the process. Hence, for solving the main problem of this paper we may assume (without loss of generality and whenever convenient) that $P\{T_x < \infty\} = 1$ for all $x > 0$ and that, for example, the process X has no jumps larger in magnitude than (say) 1.

Assume now for the rest of this section that (1.3), (1.4), (1.5) hold, and that one of the cases (c)–(f) is present. Under these hypotheses it is known (Bretagnolle [6]) that 0 is regular for itself so that for each real number y there is a continuous additive functional $L^y = \{L_t^y, t \geq 0\}$ called the local time at y. This additive functional is unique only up to constant multiples; we may choose L^y for each y in such a way, however, as to be jointly measurable in (t, y, ω) and to satisfy for all Borel sets B simultaneously:

$$\int_B L_t^y \, dy = \int_0^t I_B(X_s)\, ds. \tag{3.7}$$

This result is due to Blumenthal and Getoor ([2]; see also [9]).

The following proposition gives a necessary and sufficient condition that $P\{X_{T_x} = x\} > 0$ for all $x > 0$, assuming X has local times.

Proposition 3.2. *Let $X = \{X_t, t \geq 0\}$ be a real process with stationary independent increments having local times L^y as described above. Then $P\{X_{T_x} = x\} > 0$ for all $x > 0$ if and only if $\int_0^1 E^0 L^0_{T_y} \nu[y]\, dy < \infty$, where $\nu[y] = \nu\{(y, \infty)\}$.*

Proof. Without loss of generality, assume ν has no mass outside the interval $(-1, 1)$. Then by (3.7) and Proposition 2.5:

$$\begin{aligned} P\{X_{T_x} > x\} &= E^0 \int_{y=0}^{\infty} \int_{s=0}^{T_x} I_{(x-y,x)}(X_s)\, ds\,\nu(dy) \\ &= E^0 \int_{y=0}^{\infty} \int_{x-y}^{x} L^u_{T_x}\, du\,\nu(dy) = E^0 \int_{u=-\infty}^{x} \int_{y=x-u}^{\infty} L^u_{T_x}\, \nu(dy)\, du \\ &= E^0 \int_{-\infty}^{x} L^u_{T_x}\, \nu[x-u]\, du = E^0 \int_0^{\infty} L^{x-y}_{T_x} \nu[y]\, dy \\ &= \int_0^1 P^0\{\text{hit } x - y \text{ before exceeding } x\}\, E^0 L^0_{T_y}\, \nu[y]\, dy \end{aligned} \tag{3.8}$$

since $\nu[y] = 0$ if $y > 1$. Let $J_t = X_t - X_{t-}$. Using the same argument as in Proposition 2.5 (with $f(u, v) = v - u$ if $u < x < v$, $f(u, v) = 0$ otherwise) we find

$$E^0 J_{T_x} = E^0 \int_{y=0}^{\infty} \int_{s=0}^{T_x} y I_{(x-y,x)}(X_s)\, ds\,\nu(dy); \tag{3.9}$$

and if $V[y] = \int_y^\infty u\nu(du) = \int_y^1 u\nu(du)$, (3.9) reduces under the present hypotheses to

$$E^0 J_{T_x} = \int_0^1 P^0\{\text{hit } x - y \text{ before exceeding } x\} E^0 L^0_{T_y} V[y]\, dy. \tag{3.10}$$

Also under the present hypotheses, $\lim_{x\downarrow 0} T_x = 0$ and $0 \le J_{T_x} \le 1$; hence,

$$\lim_{x\downarrow 0} E^0 J_{T_x} = 0 \tag{3.11}$$

by the dominated convergence theorem. Note that $0 < E^0 L^0_{T_y} \le E^0 L^0_{T_1} < \infty$, $0 < y < 1$; $0 < V[y]$, for $0 < y \le a$, some $a > 0$; and $\int_0^1 V[y]\,dy < \infty$. For each x, let $f_x(\cdot)$ be the function $f_x(y) = P\{\text{hit } x - y \text{ before exceeding } x\}$. It then follows from (3.11) that the family of functions f_x converges to 0 in measure (for Lebesgue measure on $[0, a]$) as $x \downarrow 0$.

Now assume that $\int_0^1 E^0 L^0_{T_y} \nu[y]\,dy < \infty$. Then since $\{f_x\}$ converges in measure as $x \downarrow 0$, it follows from (3.8) that $\lim_{x\downarrow 0} P\{X_{T_x} > x\} = 0$. An application of Corollary 3.3 completes the proof in this case. (One may also complete this part of the proof by using the argument at the end of Theorem 2.2, and thus avoid the use of the subordinator Y). Conversely, assume that $P\{X_{T_x} = x\} > 0$. Then $g(x) = P\{X_{T_x} = x\}$ is continuous, positive, and $\lim_{x\downarrow 0} g(x) = 1$, so that $g(x) \ge \delta > 0$ for all $x \in (0, \epsilon)$, say. Also, for $y > 0$, $P^0\{\text{hit } x - y \text{ before exceeding } x\} \ge P^0\{X_{T_{x-y}} = x - y\} = g(x - y)$. But then by (3.8)

$$1 \ge P\{X_{T_\epsilon} > \epsilon\} \ge \delta \int_0^\epsilon E^0 L^0_{T_y} \nu[y]\,dy$$

which implies that $\int_0^1 E^0 L^0_{T_y} \nu[y]\,dy < \infty$.

Proposition 3.2 gives a quick solution to case (d).

Theorem 3.1. *Suppose that X satisfies (1.3) and that $\int_0^1 x\nu(dx) < \infty$, $\int_{-1}^0 |x|\nu(dx) = \infty$. Then $P\{X_{T_x} = x\} > 0$ for all $x > 0$.*

Notice that the proof works whether or not there is a Gaussian component.

Proof. $\int_0^1 EL^0_{T_y} \nu[y]\,dy \le EL^0_{T_1} \int_0^1 \nu[y]\,dy < \infty$, since the hypothesis $\int_0^1 x\nu(dx) < \infty$ implies $\int_0^1 \nu[y]\,dy < \infty$.

Next, define

$$H_x = \inf\{t > 0\colon X_t = x\}. \tag{3.12}$$

Obviously $H_x \ge T_x$. The next proposition gives an improvement of Proposition 3.2; the improvement lies in the fact that $EL^0_{H_x}$ happens to be more easily analysed than $EL^0_{T_x}$, as we will see below.

Proposition 3.3. *Let X have local times as described at the beginning of this section. Then $P\{X_{T_x} = x\} > 0$ for all $x > 0$ if and only if $\int_0^1 EL^0_{H_y} \nu[y]\,dy < \infty$.*

Proof. If $\int_0^1 EL^0_{H_y} \nu[y]\,dy < \infty$, then since $EL^0_{H_y} \ge EL^0_{T_y}$, it is evident that $\int_0^1 EL^0_{T_y} \nu[y]\,dy < \infty$. Proposition 3.2 then guarantees that $P\{X_{T_x} = x\} > 0$ for all $x > 0$.

Conversely, suppose that $P\{X_{T_x} = x\} > 0$ for all $x > 0$. Since L^0 increases "only" when X visits 0 (see [3, V.3] for the precise result) and because of the strong Markov property, we have

$$EL^0_{H_y} = EL^0_{T_y} + P^0\{T_y < H_y \text{ and } X \text{ hits 0 between } T_y \text{ and } H_y\}EL^0_{H_y}$$
$$\leq EL^0_{T_y} + P^0\{T_y < H_y\}EL^0_{H_y}.$$

But if $P\{X_{T_x} = x\} > 0$ for every $x > 0$, then by Corollary 3.2, $\lim_{y\downarrow 0} P\{T_y < H_y\} = 0$ ($\{H_x = T_x\} = \{X_{T_x} = x\}$). Hence there exists $\delta > 0$ such that $P\{T_y < H_y\} \leq ½$, $0 < y \leq \delta$, so that

$$EL^0_{H_y} \leq 2EL^0_{T_y}, \qquad 0 \leq y \leq \delta. \tag{3.13}$$

Also, if $P\{X_{T_x} = x\} > 0$ for all $x > 0$, then $\int_0^1 EL^0_{T_y}\nu[y]dy < \infty$, so $\int_0^1 EL^0_{H_y}\nu[y]dy < \infty$ from (3.13), as desired.

The next proposition, which is due to Kesten, is the first step in reducing the criterion of Proposition 3.3 to a tractable analytic expression. A different proof of this proposition appears in [12]; the present simpler proof is also due to Kesten.

Proposition 3.4 (Kesten). *Let X be a process with stationary independent increments having a local time at 0. Let $u^\lambda(x) = E^x \int_0^\infty e^{-\lambda t} d_t L^0_t$. Then*

$$EL^0_{H_y} = \lim_{\lambda\downarrow 0} u^\lambda(0)[1 - Ee^{-\lambda H_y}Ee^{-\lambda H_{-y}}]. \tag{3.14}$$

Remark. Observe that according to this proposition $EL^0_{H_y} = EL^0_{H_{-y}}$, a fact that will be useful later.

Proof. Let $\lambda > 0$. Then using the strong Markov property:

$$u^\lambda(0) = E^0 \int_0^\infty e^{-\lambda t} dL^0_t$$

$$= E^0 \int_0^{H_y} e^{-\lambda t} dL^0_t + E^0 e^{-\lambda H_y} E^y \int_0^\infty e^{-\lambda t} dL^0_t$$

$$= E^0 \int_0^{H_y} e^{-\lambda t} dL^0_t + E^0 e^{-\lambda H_y} E^y e^{-\lambda H_0} E^0 \int_0^\infty e^{-\lambda t} dL^0_t$$

$$= E^0 \int_0^{H_y} e^{-\lambda t} dL^0_t + E^0 e^{-\lambda H_y} E^y e^{-\lambda H_0} u^\lambda(0).$$

Hence,

$$E^0 \int_0^{H_y} e^{-\lambda t}\, dL_t^0 = u^\lambda(0)[1 - E^0 e^{-\lambda H_y} E^y e^{-\lambda H_0}]$$

$$= u^\lambda(0)[1 - E^0 e^{-\lambda H_y} E^0 e^{-\lambda H_{-y}}].$$

Since $E^0 \int_0^{H_y} e^{-\lambda t}\, dL_t^0 \uparrow E^0 L_{H_y}^0$ as $\lambda \downarrow 0$, the proof is complete.

Using Proposition 3.4 we next obtain an analytic necessary and sufficient condition for $P\{X_{T_x} = x\} > 0$ for all $x > 0$.

Theorem 3.2. *Let X have local times as described earlier in this section. Then $P\{X_{T_x} = x\} > 0$ for all $x > 0$ if and only if $\int_0^1 \nu[y] K(y)\, dy < \infty$, where $\nu[y] = \nu\{(y, 1)\}$ and*

$$K(y) = \lim_{\lambda \downarrow 0} \int_{-\infty}^{\infty} [1 - \cos uy] \operatorname{Re}\{(\lambda - \psi(u))^{-1}\} du.$$

Proof. From Proposition 3.3, it is enough to show that

$$EL_{H_y}^0 \sim \pi^{-1} K(y) \quad \text{as } y \downarrow 0. \tag{3.15}$$

Since 0 is regular, Theorem 2 of Kesten's monograph [11, p. 7] guarantees that

$$\int_{-\infty}^{\infty} \operatorname{Re}(\lambda - \psi(u))^{-1}\, du < \infty \quad \text{for all } \lambda > 0. \tag{3.16}$$

Moreover, according to Bretagnolle [6] there is for each $\lambda > 0$ a bounded continuous density $u^\lambda(x)$ such that

$$E \int_0^\infty e^{-\lambda t} I_A(X_t)\, dt = \int_A u^\lambda(x)\, dx \quad \text{for every Borel set } A \tag{3.17}$$

and also

$$u^\lambda(x) = u^\lambda(0) E e^{-\lambda H_x}. \tag{3.18}$$

This density $u^\lambda(\cdot)$ is, of course, the same object that appears in Proposition 3.4 (consult [9] for more detail on u^λ). Using the inversion formula for characteristic functions (see [7, Theorem 6.2.1]), (3.16) and (3.17) we find

$$\int_{-y}^{y} u^\lambda(z)\, dz = \int_0^\infty e^{-\lambda t} P\{|X_t| \le y\}\, dt$$

$$= \pi^{-1} \int_{-\infty}^{\infty} u^{-1} \sin uy \operatorname{Re}(\lambda - \psi(u))^{-1}\, du \tag{3.19}$$

$$= \pi^{-1} \int_0^y dz \int_{-\infty}^{\infty} \cos uz \operatorname{Re}(\lambda - \psi(u))^{-1}\, du,$$

implying that

$$u^\lambda(y) + u^\lambda(-y) = \pi^{-1} \int_{-\infty}^{\infty} \cos uy \operatorname{Re}(\lambda - \psi(u))^{-1}\, du, \tag{3.20}$$

(3.21) $$u^{\lambda}(0) = (2\pi)^{-1} \int_{-\infty}^{\infty} \operatorname{Re}(\lambda - \psi(u))^{-1}\, du.$$

(The calculation (3.19) is in [9] and [12].) Next

$$\begin{aligned} u^{\lambda}(0)[1 - Ee^{-\lambda H_y} Ee^{-\lambda H_{-y}}] &= u^{\lambda}(0) - u^{\lambda}(y) Ee^{-\lambda H_{-y}} \\ &= u^{\lambda}(0) - u^{\lambda}(y) + Ee^{-\lambda H_y}[u^{\lambda}(0) - u^{\lambda}(-y)] \\ &\leq 2u^{\lambda}(0) - u^{\lambda}(y) - u^{\lambda}(-y) \\ &= \pi^{-1} \int_{-\infty}^{\infty} [1 - \cos uy] \operatorname{Re}(\lambda - \psi(u))^{-1}\, du, \end{aligned}$$

using the fact that $u^{\lambda}(0) \geq u^{\lambda}(x)$ by (3.18). On the other hand, $u^{\lambda}(x) \leq u^{\lambda}(0)$ and $H_y \to 0$ in probability as $y \to 0$ (Bretagnolle [6]). Hence for all sufficiently small y (how small depends on ϵ and not on $\lambda \in (0, 1)$):

$$u^{\lambda}(0)[1 - Ee^{-\lambda H_y} Ee^{-\lambda H_{-y}}] \geq (1 - \epsilon)\pi^{-1} \int_{-\infty}^{\infty} (1 - \cos uy) \operatorname{Re}(\lambda - \psi(u))^{-1}\, du$$

for $0 < \epsilon < 1$, so that

$$EL^0_{H_y} \sim \lim_{\lambda \downarrow 0} \pi^{-1} \int_{-\infty}^{\infty} (1 - \cos uy) \operatorname{Re}(\lambda - \psi(u))^{-1}\, du$$

as desired.

In certain cases of common occurrence, one can evaluate $K(y)$ more conve-iently by taking the limit under the integral.

Corollary 3.5. *If* $\liminf_{|u| \to \infty} \operatorname{Re}\{-\psi(u)\} \geq \delta > 0$, *then*

$$EL^0_{H_y} \sim \pi^{-1} K(y) = \pi^{-1} \int_{-\infty}^{\infty} [1 - \cos uy] \operatorname{Re}\{-\psi(u)\} |\lambda - \psi(u)|^{-2}\, du.$$

This hypothesis is satisfied if the measure corresponding to the characteristic function $\exp \psi(u)$ *is not purely singular relative to Lebesgue measure.*

Proof.

$$\begin{aligned} \int_{-\infty}^{\infty} (1 - \cos uy) \operatorname{Re}(\lambda - \psi(u))^{-1}\, du &= \int_{-\infty}^{\infty} (1 - \cos uy) \operatorname{Re}\{-\psi(u)\} |\lambda - \psi(u)|^{-2}\, du \\ &+ \lambda \int_{|u|<1} (1 - \cos uy) |\lambda + \psi(u)|^{-2}\, du \\ &+ \lambda \int_{|u| \geq 1} (1 - \cos uy) |\lambda + \psi(u)|^{-2}\, du. \end{aligned}$$

By monotone convergence,

$$\int_{-\infty}^{\infty} (1 - \cos uy) \operatorname{Re}\{-\psi(u)\} |\lambda + \psi(u)|^{-2}\, du$$

$$\to \int_{-\infty}^{\infty} (1 - \cos uy) \operatorname{Re}\{-\psi(u)\} |\psi(u)|^{-2}\, du \leq \pi EL^0_{H_y} < \infty$$

as $\lambda \downarrow 0$; while, since $|\psi(u)| \geq \text{const } u^2$ for $|u| \leq 1$,

$$\lambda \int_{|u|\leq 1} (1 - \cos uy)|\lambda + \psi(u)|^{-2}\, du \to 0.$$

Finally, (this is the only place where the hypothesis is used):

$$\lambda \int_{|u|\geq 1} (1 - \cos uy)|\psi(u) + \lambda|^{-2}\, du \leq \lambda \int_{|u|\geq 1} (1 - \cos uy)|\psi(u)|^{-2}\, du$$

$$\leq \lambda\delta^{-1} \int_{|u|\geq 1} (1 - \cos uy)\, \mathrm{Re}\{-\psi(u)\}\, |\psi(u)|^{-2}\, du$$

$$\leq \pi\lambda\delta^{-1} EL^0_{H_y} \to 0,$$

using the first part of the proof. This completes the proof since the implication of the second sentence of the corollary is well known. I do not know whether the hypothesis of the corollary can be removed.

We will now apply the criterion of Theorem 3.2 to several cases of particular interest. Let us first consider the case when X has a Gaussian component.

Theorem 3.3. *Let X be a process with stationary independent increments having $\sigma^2 > 0$. Then $P\{X_{T_x} = x\} > 0$ for all $x > 0$.*

Proof. Using Corollary 3.5 we will verify that if $\sigma^2 > 0$, then $EL^0_{H_y} \leq$ const y for all sufficiently small y. The conclusion will then follow from Theorem 3.2, since

$$\int_0^1 EL^0_{H_y} \nu[y]\, dy \leq \text{const} \int_0^1 y\nu[y]\, dy = \text{const} \int_0^1 y^2 \nu(dy) < \infty.$$

However, from (3.15) and an elementary inequality

$$EL^0_{H_y} \leq \pi^{-1} \int_{-\infty}^{\infty} (1 - \cos uy)[\mathrm{Re}\{-\psi(u)\}]^{-1}\, du$$

$$\leq \pi^{-1} \int_{-\infty}^{\infty} (1 - \cos uy)[(\sigma^2/2)u^2]^{-1}\, du$$

$$= y 4\pi^{-1}\sigma^{-2} \int_{-\infty}^{\infty} (1 - \cos z) z^{-2}\, dz,$$

as desired. We have used in this calculation the fact that the real part of the exponent of a process is negative.

In order to state the next result it is convenient to have some further notation. If ν is the Lévy measure of the process X, let ν_+ be ν restricted to $(0, \infty)$ and ν_- be ν restricted to $(-\infty, 0)$. Assuming that X has no Gaussian component, the next intuitively appealing result says essentially that if ν_- is rather larger than ν_+, then X will have continuous passages upward (i.e. $P\{X_{T_x} = x\} > 0$ for all $x > 0$). In order to give a precise meaning to this we introduce the indices

β, β'' of Blumenthal and Getoor [1]: let X be a process with no Gaussian part having exponent ψ and Lévy measure ν. Define

(3.22) $$\beta(X) = \inf\left\{\alpha \geq 0 : \lim_{|y|\to\infty} |y|^{-\alpha}\operatorname{Re}\{-\psi(y)\} = 0\right\},$$

(3.23) $$\beta''(X) = \sup\left\{\alpha \geq 0 : \lim_{|y|\to\infty} |y|^{-\alpha}\operatorname{Re}\{-\psi(y)\} = \infty\right\}.$$

For a given process X, let X^+ be the process with Lévy measure ν_+, and let ψ_+ be the exponent of X^+:

$$\psi_+(u) = \int_{-\infty}^{\infty} [e^{iux} - 1 - iux/(1 + x^2)]\nu_+(dx)$$

$$= \int_0^{\infty} [e^{iux} - 1 - iux/(1 + x^2)]\nu(dx).$$

Let $X^- = X - X^+$ and let ψ_- be the exponent of X^-. Then we have the following theorem.

Theorem 3.4. *Assume* $\beta''(X^-) > 1$, *and* $\beta''(X^-) > \beta(X^+)$. *Then* $P\{X_{T_x} = x\} > 0$ *for all* $x > 0$.

Remarks. Examples of such processes may be constructed as follows. Let X' be a stable process with index α' and Lévy measure concentrated on $(0, \infty)$, and X'' a stable process, independent of X', with index α'' and Lévy measure on $(-\infty, 0)$. If $\alpha'' > 1$ and $\alpha'' > \alpha'$, then $X' + X''$ is an example of the processes described in Theorem 3.4. According to Theorem 3.5 these processes will not have continuous passages in the negative direction.

Proof. From Corollary 3.5, $EL^0_{H_y} \leq \pi^{-1}\int_{-\infty}^{\infty}(1 - \cos uy)(\operatorname{Re}\{-\psi_-(u)\})^{-1}du$. Since for $|u| \leq 1$, $\operatorname{Re}\{-\psi_-(u)\} \geq \text{const } u^2$,

$$\pi^{-1}\int_{|u|\leq 1} (1 - \cos uy)(\operatorname{Re}\{-\psi_-(u)\})^{-1} \leq \text{const } y^2.$$

Choose β subject to $\beta''(X^-) > \beta > 1$, $\beta > \beta(X^+)$. Then for some constant A, $\operatorname{Re}\psi_-(u) \geq Au^{\beta}$, $|u| \geq 1$, implying that

$$\pi^{-1}\int_{|u|>1} (1 - \cos uy)(\operatorname{Re}\{-\psi_-(u)\})^{-1} \leq \text{const}\int_{|u|>1} (1 - \cos uy)u^{-\beta}du$$

$$= \text{const } y^{\beta-1}.$$

Hence $EL^0_{H_y} \leq \text{const } y^{\beta-1}$. Let γ be chosen to satisfy $\beta > \gamma > \beta(X^+)$. By Theorem 2.1 of [1], if $y > 0$, then $\nu_+[y] = \nu[y] \leq \text{const } y^{-\gamma}$. $\int_0^1 EL^0_{T_y}\nu[y]dy \leq \text{const}\int_0^1 y^{\beta-1}y^{-\gamma}dy < \infty$ and the result follows from Theorem 3.2.

The remaining results of this section are perhaps most easily deduced by using a recent result of Fristedt [8]. Recall the definitions of the subordinators $\{\tau_t, t \geq 0\}$ and $\{Y_t, t \geq 0\}$ from (3.1) and (3.2). Fristedt has then shown that

$$E \exp\{-\lambda_1 \tau_t - \lambda_2 Y_t\} = \exp\{-t\psi(\lambda_1, \lambda_2)\} \tag{3.24}$$

where, if F_t is the distribution of X_t,

$$\psi(\lambda_1, \lambda_2) = \exp\left\{\int_{t=0}^{\infty} t^{-1}\, dt \int_{x=0}^{\infty} [e^{-\lambda_1 t} - e^{-\lambda_1 t - \lambda_2 x}] F_t(dx)\right\}. \tag{3.25}$$

It then follows that $\{Y_t\}$ will have positive drift if and only if

$$\lim_{\lambda_2 \to \infty} \lambda_2^{-1} \psi(1, \lambda_2) = c > 0;$$

that is, $\{Y_t\}$ has positive drift if and only if

$$\int_0^{\infty} t^{-1}\, dt \int_0^{\infty} [e^{-t} - e^{-t-\lambda x}] F_t(dx) - \log \lambda \to c_2 > -\infty \tag{3.26}$$

as $\lambda \to \infty$.

Recall the definition $T_{-x} = \inf\{t > 0: X_t < -x\}$, $x > 0$. The next result says that the only processes that have continuous passages both upwards and downwards are those processes having a Gaussian component.

Theorem 3.5. *Suppose X is a process with stationary independent increments for which $P\{X_{T_x} = x\} > 0$ and $P\{X_{T_{-x}} = -x\} > 0$ for all $x > 0$. Then X must have a Gaussian component.*

The converse to this theorem was established in Theorem 3.3.

Proof. Let $Y = \{Y_t\}$ be the subordinator defined in (3.2) and let $Y' = \{Y_t'\}$ be the analogous subordinator for the process $\{-X_t\}$. By Corollary 3.1, Y and Y' both must have positive drift. From the criterion (3.26) both (i) and (ii) hold as $\lambda \to \infty$:

(i) $\int_0^\infty e^{-t} t^{-1}\, dt \int_0^\infty [1 - e^{-\lambda x}] F_t(dx) - \log \lambda \to c_3 > -\infty$,

(ii) $\int_0^\infty e^{-t} t^{-1}\, dt \int_{-\infty}^0 [1 - e^{-\lambda |x|}] F_t(dx) - \log \lambda \to c_4 > -\infty$,

so that

$$\int_0^{\infty} e^{-t} t^{-1}\, dt \int_{-\infty}^{\infty} [1 - e^{-\lambda |x|}] F_t(dx) - \log(\lambda^2) \to C > -\infty. \tag{3.27}$$

However, by the first part of the proof of Theorem 2.1 of [17],

$$\int_{-\infty}^{\infty} [1 - e^{-\lambda |x|}] F_t(dx) = \pi^{-1} \int_{-\infty}^{\infty} [1 - e^{-t\psi(\lambda y)}](1 + y^2)^{-1}\, dy.$$

Moreover, if z is a complex number, $\operatorname{Re} z > 0$, then

$$\operatorname{Re} \int_0^{\infty} e^{-t} t^{-1} (1 - e^{-tz})\, dt = \log |z + 1|. \tag{3.28}$$

Hence, from (3.27) and (3.28),

$$\lim \pi^{-1} \int_{-\infty}^{\infty} \log \{|1 - \psi(\lambda y)|\lambda^{-2}\}(1 + y^2)^{-1}\, dy = c > -\infty. \tag{3.29}$$

By a well-known result (possibly first noticed by Bochner [4, Theorem 3.4.2]), $\lim_{\lambda\to\infty}|\psi(\lambda y)|\lambda^{-2} = 0$ if and only if there is no Gaussian part. Hence (3.29) can hold only if there is a Gaussian part, and this is what we wanted to prove.

We can now easily dispose of case (e) of (1.7).

Theorem 3.6. *Let X be a process with stationary independent increments having $\sigma^2 = 0$, $\int_0^1 x\nu(dx) = \infty$, $\int_{-1}^0 |x|\nu(dx) < \infty$. Then $P\{X_{T_x} = x\} = 0$ for all $x > 0$.*

Proof. By Theorem 3.1, $P\{X_{T_{-x}} = -x\} > 0$ for all $x > 0$. Since X has no Gaussian part, Theorem 3.4 implies that $P\{X_{T_x} = x\} = 0$ for all $x > 0$.

We conclude this section with the following useful criterion.

Theorem 3.7. *Let X be a process with stationary independent increments having no Gaussian component but admitting a local time at 0. Assume that $\nu\{(-\infty, -x)\} = O(\nu\{(x, \infty)\})$ as $x \downarrow 0$. Then $P\{X_{T_x} = x\} = 0$ for all $x > 0$.*

Proof. Let $\nu_+[y] = \nu\{(y, \infty)\}$, $\nu_-[y] = \nu\{(-\infty, -y)\}$ for $y > 0$. According to Theorem 3.5 and Proposition 3.3 at least one of the following must occur:

(i) $\int_0^1 EL^0_{H_y}\nu_+[y]dy = \infty$,

(ii) $\int_0^1 EL^0_{H_{-y}}\nu_-[y]dy = \infty$.

By Proposition 3.4, $EL^0_{H_{-y}} = EL^0_{H_y}$ so that (ii) is the same as

(ii)′ $\int_0^1 EL^0_{H_y}\nu_-[y]dy = \infty$.

Since at least one of (i), (ii)′ must hold and since by hypothesis ν_+ is bigger than ν_-, clearly (i) must hold. Hence by Proposition 3.3, $P\{X_{T_x} = x\} = 0$ for all $x > 0$.

The following special case is worth pointing out.

Corollary 3.5. *Suppose X has no Gaussian component. If $\nu\{(-\infty, -x)\} = O(\nu\{(x, \infty)\})$ and if $\nu\{(x, \infty)\} = O(\nu\{(-\infty, -x)\})$ as $x \downarrow 0$, then $P\{X_{T_x} = x\} = P\{X_{T_{-x}} = -x\} = 0$ for all $x > 0$.*

In particular any symmetric process and any stable process that does not have its Lévy measure concentrated on a half-axis will have continuous passages in neither the upward nor downward direction.

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